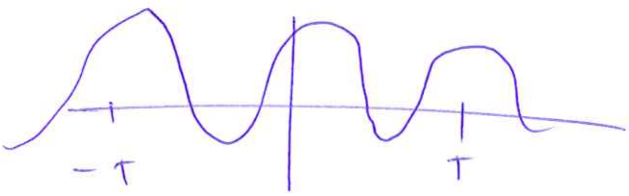
Fourier series

Every periodic continuous-time signal can be written as a sum of sinusoids

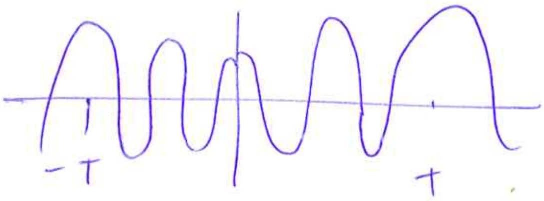
$$x(t+T) = x(t)$$

Are the signals that have period T ?
other

→ cos



$\cos \omega_0 t = \cos \frac{2\pi}{T} t$



$\cos 2\omega_0 t = \cos \frac{4\pi}{T} t$

→ $\cos k\omega_0 t$, k is int

→ complex signal

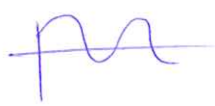
$e^{jk\omega_0 t} = \cos k\omega_0 t + j \sin k\omega_0 t$ thanks to euler formula

similary $\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$ is also periodic with period T

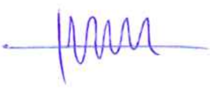
↳ coeff that change Amplitude, (any complex number)

we would do

$\cancel{X(k) = \int_{-\infty}^{\infty} x(t) \cos \frac{2\pi}{T} \omega_0 k t \quad \sin \omega_0 t \quad x_e(t)}$

$= a_1 \cdot$ 

$+ a_2 \cdot$ 

$+ a_3 \cdot$ 

GOAL : represent $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$

→ fourier series

How to find $\{a_k\}$'s ? After integrating both sides :
(formulas not shown)

$$a_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt$$

$\{a_k\}$ are the fourier series coeff of $x(t)$

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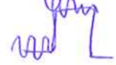
demo applet (fourier series applet on falstad.com)

- squares : more terms, best fit
  a_k's decreasing : normal : a_5 responsible for $5 \times \omega_0 t$
  (magnitude)

we don't want crazy wiggles → a_5 is small

function of square is not continuous, ~~(not differentiable)~~
→ there is a shoulder



→ triangle : looks like sin, compute first $\{a_k\}$ and it fits

→ sawtooth : same shoulder because not continuous

→ square : phase shift button
→ magnitude doesn't change but phase does

→ same with cosines and sine

→ GIBBS : we can't go below 9% of the height

Fourier transform

Discrete

	continuous	discrete
periodic	FS	DFT / FFT
aperiodic	FT	DTFT

DFT samples the DTFT in the frequency domain

DTFT évalue la z-transform de $X(\omega)$ pour un nombre infini de valeurs ω

FS

$$x(t) = \sum_{k=-\infty}^{\infty} \underset{(a_k)}{X(k)}\, e^{jk\overbrace{\frac{2\pi}{T}}^{\omega_0 \text{ (fréquence fondamentale)}} t}$$

if $x(t)$ wiggles infinitely fast, need an infinite number of a_k

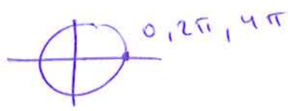
let's try

$$x(n) = \sum_{k=-\infty}^{\infty} X(k)\, e^{jk\frac{2\pi}{N}n} \qquad n = 0, 1, \dots, N-1$$

doesn't work : discrete n and summing an infinite number of k values?

work around

$$\to e^{jk\frac{2\pi}{N}(n+N)} = e^{jk\frac{2\pi}{N}n}\, \underbrace{e^{jk2\pi}}_{\text{always} = 1} \qquad k = \text{int}$$



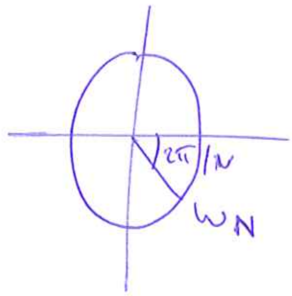
$$= e^{jk\frac{2\pi n}{N}}$$

there are only N unique complex exp of period N

$$\to X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-jk\frac{2\pi}{N}n} \qquad k = 0, \dots, N-1$$

Let's pose

$$W_N = e^{-j\frac{2\pi}{N}}$$



$$(W_N)^N = 1$$

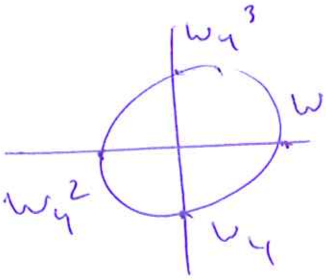
W_N is the n^{th} root of 1

We rewrite

$$X(l) = \sum_{n=0}^{N-1} x(n)\, W_N^{ln}$$

4 echantillons

W_4?



Racine 4 de 1

$$W_4 = e^{\frac{2\pi}{4}} = e^{\frac{-\pi}{2}} \qquad W_4^4 = e^{-2\pi j}$$

$$W_4^2 = e^{-\pi j} \qquad = e^{-4\pi j}$$

$$W_4^3 = e^{-\frac{3\pi}{2}j} \qquad = e^{-6\pi j}$$

$$W_4^4 = e^{-2\pi j} \qquad = e^{-8\pi j}$$