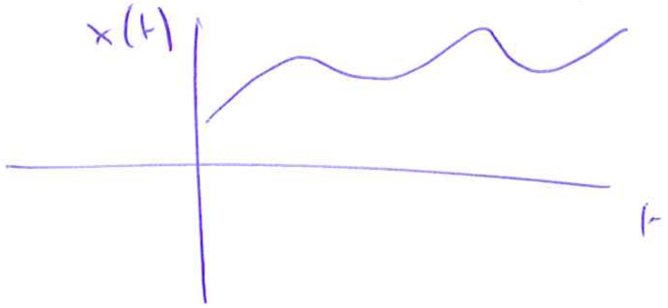
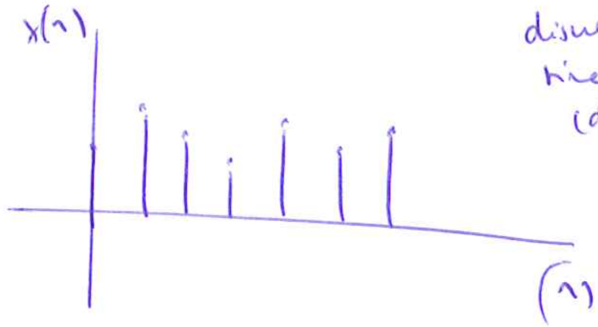
The sampling theorem

continuous (analog) time





discrete time (digital)

Periodic sampling

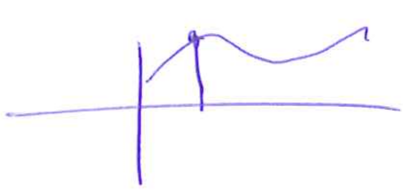
$$X(n) = x_c(nT) \quad n \text{ is int}$$

T = sampling period

$\omega_s \; \frac{2\pi}{T}$ = sampling freq. (rad)

$\frac{1}{T}$ = sampling freq (Hz)

Ideally we want the real value of the sample.



but actually

avg signal over a really short time

Non ideal effects

1) may not ideally sample $x_c(t)$
instead sample $\underbrace{x_c(t) * h(t)}_{y(t)}$ ← impulse resp of the sampler

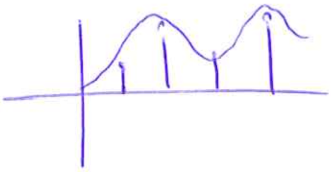
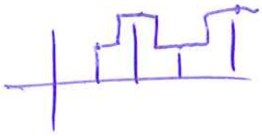
2) may have noise

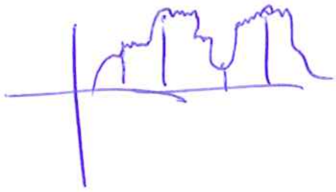
$$x(n) = y(nT) + \underbrace{Z(n)}_{\text{noise}}$$

Reconstruction

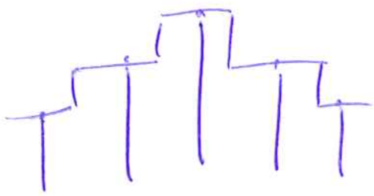
How to re-create / estimate $x_c(t)$ given $x(n)$?

so we have  should we do

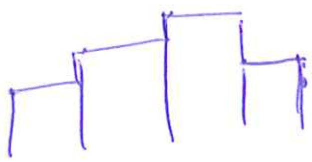
 or  no way to know...

or 

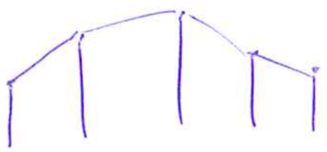
simple estimates



nearest neighbors interpolation
- take the closest sample that i see in the discrete world
  - $\rightarrow$ need to look in the future

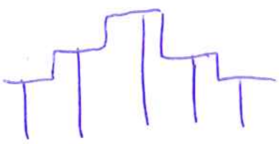
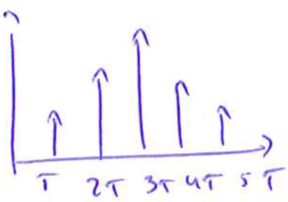


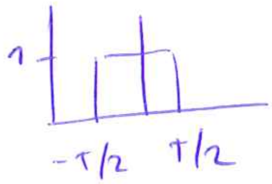
zero order hold
- use the most recent value that i saw
  - we can do better

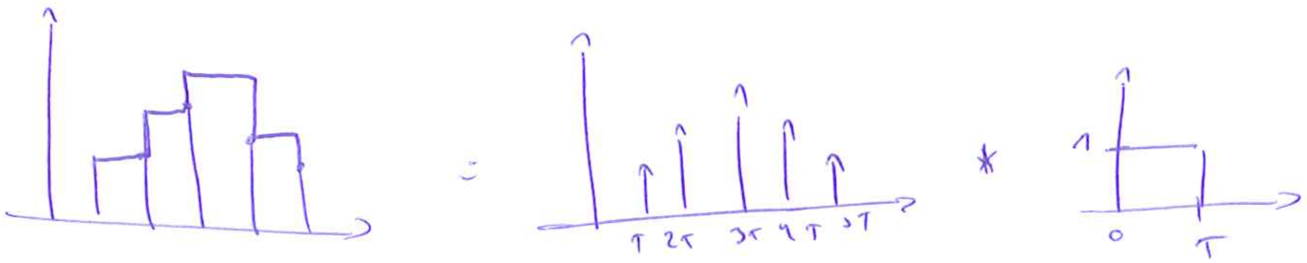


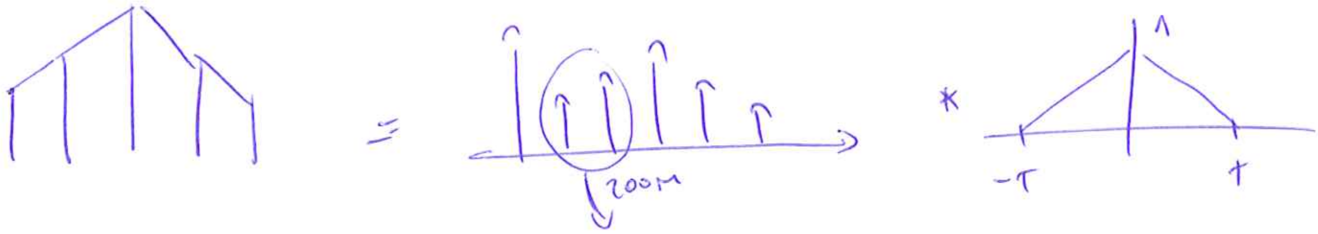
1st order hold
linear interpolation
- connecting the dots

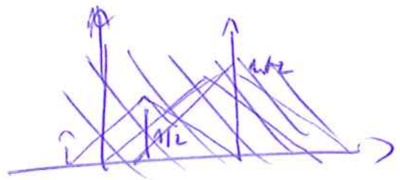
lets see them in term of the samples $x(n) * h(n)$

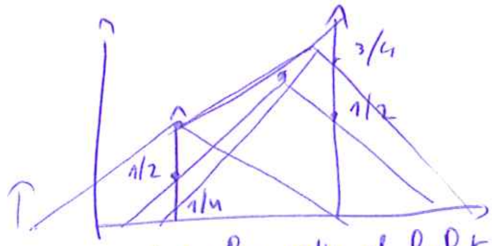
 = 

$*$ 









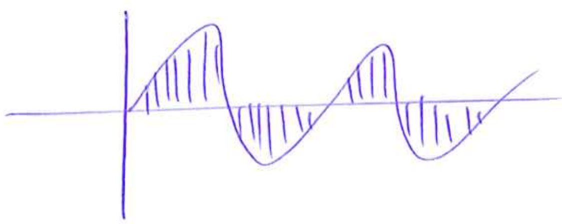
; take 1/4 of left and 3/4 of right

we weight the two adjacent samples

the actual true correct interpolation in impulse response is going to be a <u>sinc func</u> in the time domain

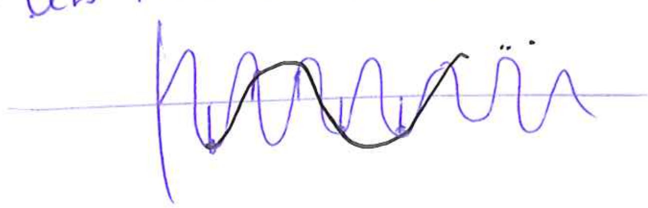
Another problem though

lets take a sinus func and sample it finely



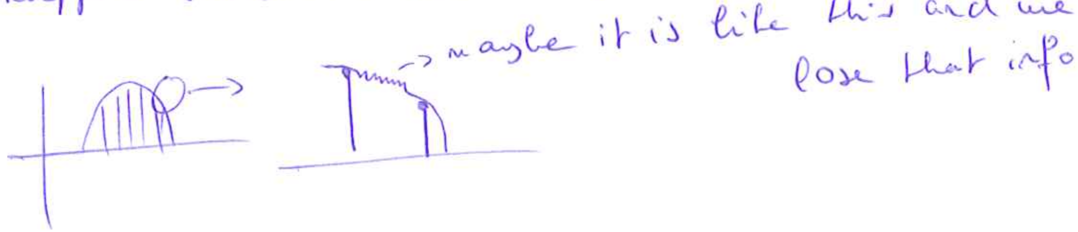
with the sample, we will reconstruct it perfectly : good case

but lets take a sin with a higher frequency.



→ cant capture all the wiggles

actually, even in the good case, who knows what really happens between two samples.



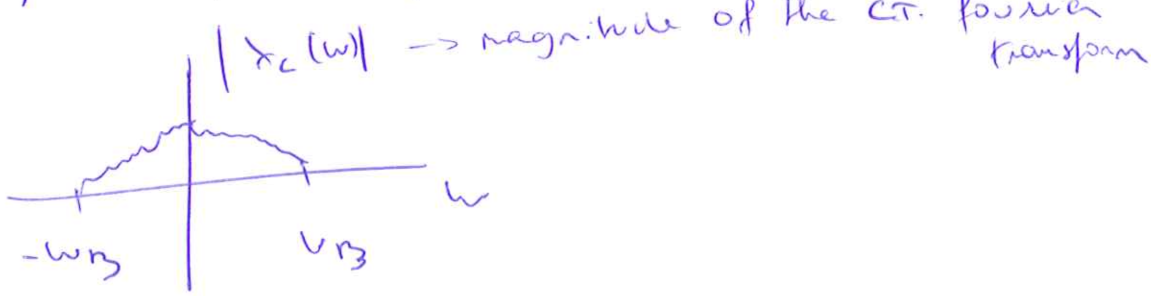
To avoid this problem, we can put restrictions on the input signal: "i do not tolerate my signal to wiggle like crazy".

To do so, we use band limited signals

Band limited signals

A signal is band-limited if there is some frequency w_B such that

$$X_c(w) = 0 \quad \text{for} \quad |w| > w_B$$



SAMPLING THEOREM

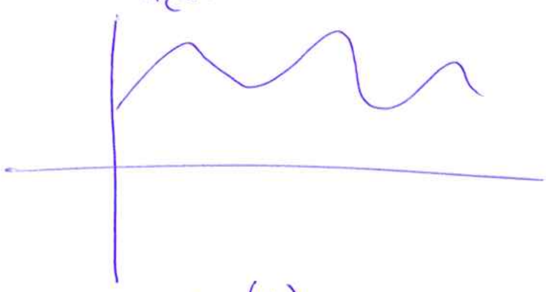
A bandlimited signal with max. freq. w_B can be reconstructed from evenly spaced samples if the sampling frequency w_s satisfies

$$w_s > 2 w_B$$

$2 w_B$ is called the nyquist rate

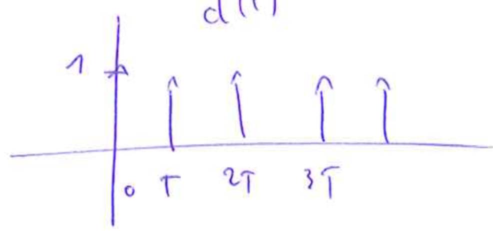
why is it true? $\qquad$ lets multiply the CT signal by the delta function (set of) $\qquad$ 5

$x_c(t)$

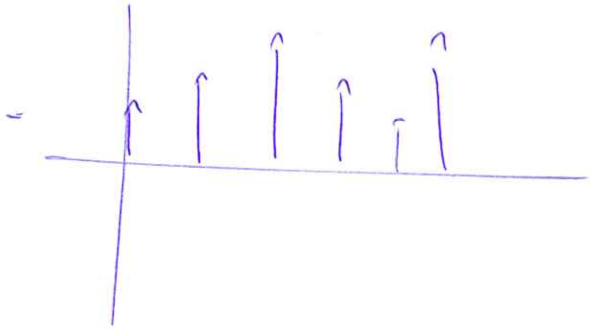


$\times$

$d(t)$



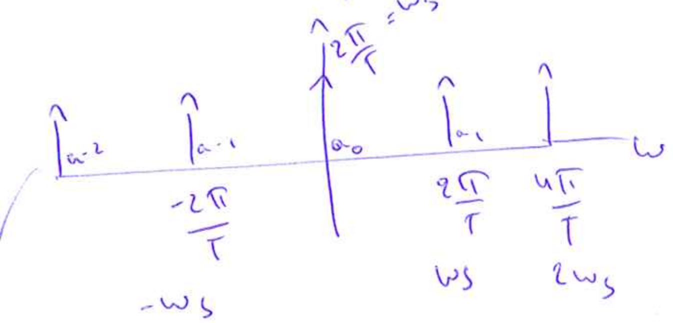
$x_s(t)$



$\rightarrow \; x_s(t) = x_c(t)\, d(t)$

To think about this in the frequency domain, we need to know what is the Fourier transform of d(t)

$D(\omega)$



these are the coeff of the fourier series

$$x_s(t) = x_c(t)\, d(t)$$

$$X_s(\omega) = \frac{1}{2\pi} X_c(\omega) * D(\omega)$$

$\Rightarrow$

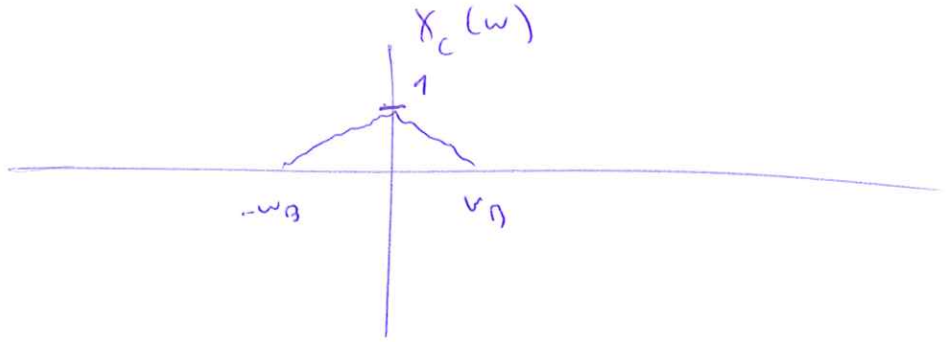
(1) $$= \frac{1}{2\pi} X_c(\omega) * \left[\frac{2\pi}{T} \sum_{-\infty}^{\infty} \delta(\omega - k\omega_s)\right]$$

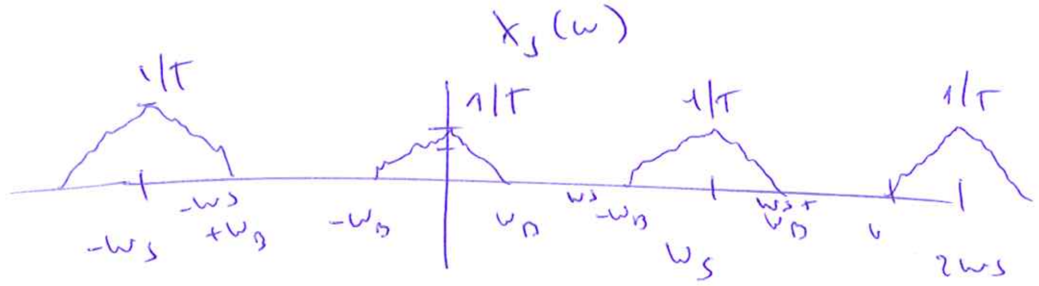
(2) $$= \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(\omega - k\omega_s)$$

(1) $\frac{2\pi}{T}$ multiplied by the sum of δ functions that are spaced out every multiples of the sampling freq

(2) when we convolve the signal with the delta function, it shifts the signal. We get a bunch of copies of the original signal at the place where the δ func fires.

lets interpret $\frac{1}{T}\sum_{l=-\infty}^{\infty} x_c(w - l w_s)$



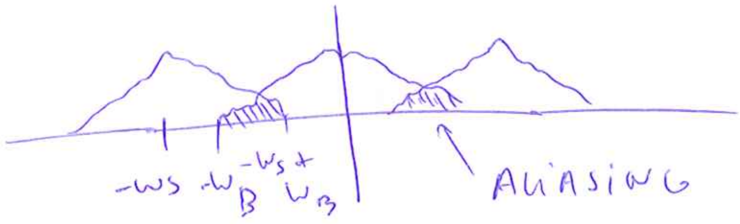


we want these two to not overlap

$\longrightarrow \quad w_s - w_B > w_B$

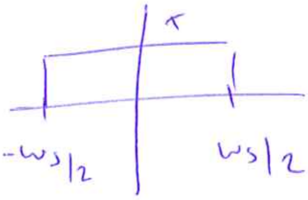
$$w_s > 2w_B$$

bad case

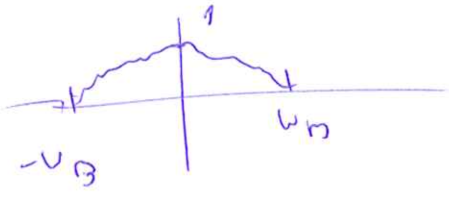


It means that if I sample too slowly, ~~wha we mix~~ the garble up high frequencies in the original signal and get distort mixed up with lower freq of the lower copy in the freq domain the ideal reconstruction filter in the ~~time~~ freq- domain is a pulse. 7

$H_r(\omega)$

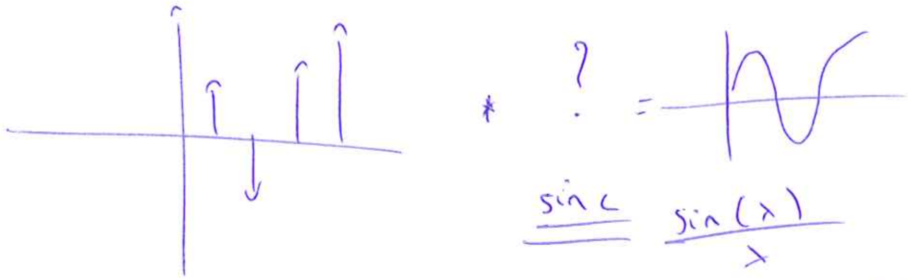


$X_s(\omega) \cdot H_r(\omega)$



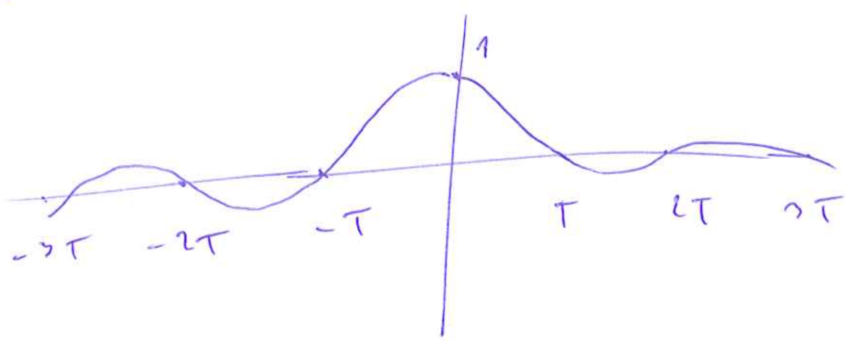
$= X_c(\omega)$

We know that a pulse in the freq domain equals / correspond to a sinc in the time domain



$\underline{\text{sinc}}$ $\frac{\sin(x)}{x}$

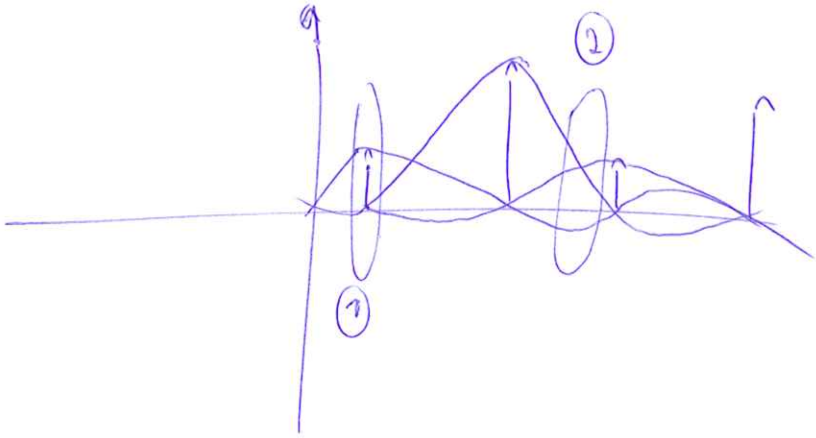
the ideal reconstructor filter in the time domain is the sinc function



$h_r(t) = \text{sinc}\left(\frac{\pi t}{T}\right)$

$$x(t) = x_s(t) * h_r(t)$$

$$= \sum_{n=-\infty}^{\infty} x(n) \, \mathrm{sinc}\left(\frac{\pi}{T}(t - nT)\right)$$



(1) at each sample, the only sinc that is non-zero is the one centered at this sample. So it equals this very sample.

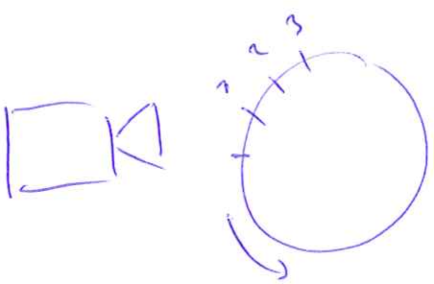
(2) the good value of the reconstructed signal happens to be the sum of all the sinc passing there.

PHASE REVERSAL

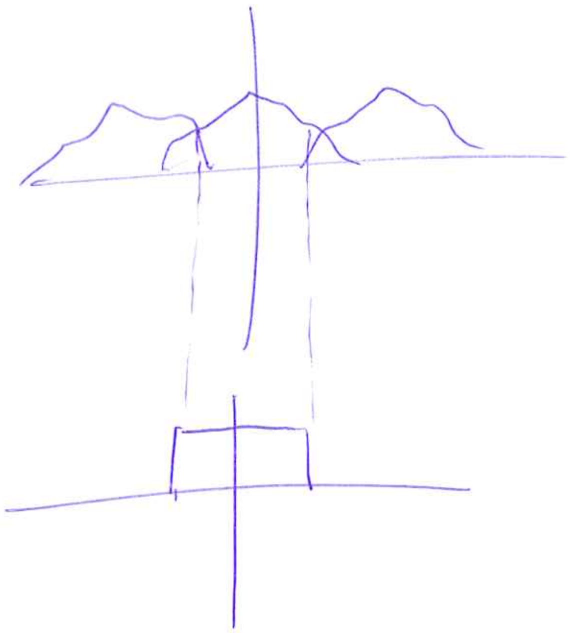
if $\omega_s < 2\omega_B$

$$x_r(t) = \cos\left((\omega_s - \omega_0)t - \underset{\text{going backwards}}{\underline{\phi}}\right)$$

$$\Rightarrow x(t) = \cos(\omega_0 t + \phi)$$



Pre filtering to avoid aliasing



we only pass what we know is okay to the sampler (we exclude the high frequencies